Properties of Exponential Functions

Capacitors are used to store electric potential energy. When a capacitor in a resistor-capacitor (RC) circuit is discharged, the electric potential across the capacitor decays exponentially over time. This sort of circuit is used in a variety of electronic devices, such as televisions, computers, and MP3 players. Engineers and technicians who design and build such devices must have a solid understanding of exponential functions.

Many situations can be modelled using functions of the form \( f(x) = ab^x \), where \( a \neq 0 \) and \( b > 0, b \neq 1 \). How do the values of \( a \) and \( b \) affect the properties of this type of function?

Investigate

How can you discover the characteristics of the graph of an exponential function?

A: The Effect of \( b \) on the Graph of \( y = ab^x \)

Start with the function \( f(x) = 2^x \). In this case, \( a = 1 \).

1. a) Graph the function.
   b) Describe the shape of the graph.

2. Use algebraic and/or graphical reasoning to justify your answers to the following.
   a) What are the domain and the range of the function?
   b) What is the \( y \)-intercept?
   c) Is there an \( x \)-intercept?
   d) Over what interval is the function
      i) increasing?
      ii) decreasing?

3. Change the value of \( b \). Use values greater than 2.
   a) Compare each graph to the graph of \( y = 2^x \). Describe how the graphs are alike. How do they differ?
   b) Describe how the value of \( b \) has affected the characteristics listed in step 2.
   c) Explain why a value of \( b \) greater than 2 has this effect on the graph.

Connections

It is important to be careful around discarded electrical equipment, such as television sets. Even if the device is not connected to a power source, stored electrical energy may be present in the capacitors.
If you are using *The Geometer’s Sketchpad®,* you can set \( b \) as a parameter whose value you can change dynamically:

- From the **Graph** menu, choose **New Parameter.** Set the name as \( b \) and its initial value to 2. Click **OK.**
- From the **Graph** menu, choose **Plot New Function.** Click on the parameter \( b \), and then click on \(^x\) and **OK.**

You can change the value of \( b \) in three ways:

- Click on parameter \( b \) and press \(+\) and \(-\) on the keyboard to increase or decrease the value of \( b \) in 1-unit increments.
- Right-click on parameter \( b \) and choose **Edit Parameter** to enter a specific value.
- Right-click on parameter \( b \) and choose **Animate Parameter.** Use the buttons on the **Motion Controller** to see the effects of changing \( b \) continuously.

4. Change the value of \( b \) again. This time, use values between 0 and 1.
   a) How has the graph changed?
   b) Describe how the values of \( b \) affect the characteristics listed in step 2.
   c) Explain why a value of \( b \) between 0 and 1 has this effect on the graph.

5. **What happens to the graph when you set** \( b = 1? \) **Explain this result.**

6. **Reflect** Summarize how the values of \( b \) affect the shape and characteristics of the graph of \( f(x) = b^x. \)

**B: The Effect of** \( a \) **on the Graph of** \( y = ab^x. \)**

Use the function \( f(x) = a \times 2^x. \) In this part of the Investigate, keep \( b = 2, \) and explore what happens as you change the value of \( a. \)

1. Set \( a = 1. \) This gives the original graph of \( f(x) = 2^x. \)
   Explore what happens when
   a) \( a > 1 \)
   b) \( 0 < a < 1 \)
   c) \( a < 0 \)

2. **Reflect** Write a summary of the effects of various values of \( a \) on the graph of the function \( f(x) = a \times 2^x. \) Include the following characteristics: domain, range, \( x-\) and \( y-\)intercepts, and intervals of increase and decrease. Explain why changing the value of \( a \) has these effects. Sketch diagrams to support your explanations.
One of the interesting features of an exponential function is its asymptotic behaviour. Consider the function \( f(x) = 2^x \). If you keep looking left at decreasing values of \( x \), you will see that the corresponding \( y \)-value of the function gets closer and closer to, but never reaches, the \( x \)-axis. In this case, the \( x \)-axis is an asymptote.

![Graph of exponential function](image)

**Example 1**

**Analyse the Graph of an Exponential Function**

Graph each exponential function. Identify the
- domain
- range
- \( x \)- and \( y \)-intercepts, if they exist
- intervals of increase/decrease
- asymptote

a) \( y = 4\left(\frac{1}{2}\right)^x \)

b) \( y = -3^{-x} \)

**Solution**

a) \( y = 4\left(\frac{1}{2}\right)^x \)

**Method 1: Use a Table of Values**

Select negative and positive values of \( x \) that will make it easy to compute corresponding values of \( y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>16</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

\[ 4\left(\frac{1}{2}\right)^2 = 4\left(\frac{2}{1}\right)^{-2} = 16 \]

Use the table of values to graph the function.

![Graph of exponential function](image)
Method 2: Use a Graphing Calculator

Use a graphing calculator to explore the graph of this function.

The function is defined for all values of \( x \). Therefore, the domain is \( \{ x \in \mathbb{R} \} \).

The function has positive values for \( y \), but \( y \) never reaches zero. Therefore, the range is \( \{ y \in \mathbb{R}, y > 0 \} \).

The graph never crosses the \( x \)-axis, which means there is no \( x \)-intercept.

The graph crosses the \( y \)-axis at 4. Therefore, the \( y \)-intercept is 4.

The graph falls to the right throughout its domain, so the \( y \)-values decrease as the \( x \)-values increase. Therefore, the function is decreasing over its domain.

As the \( x \)-values increase, the \( y \)-values get closer and closer to, but never reach, the \( x \)-axis. Therefore, the \( x \)-axis, or the line \( y = 0 \), is an asymptote.

\( b) \ y = -3^{-x} \)

The domain is \( \{ x \in \mathbb{R} \} \).

All function values are negative. Therefore, the range is \( \{ y \in \mathbb{R}, y < 0 \} \).

There is no \( x \)-intercept.

The \( y \)-intercept is –1.

The graph rises throughout its domain. Therefore, the function is increasing for all values of \( x \).

The \( x \)-axis, whose equation is \( y = 0 \), is an asymptote.
Example 2

Write an Exponential Equation Given Its Graph

Write the equation in the form $y = ab^x$ that describes the graph shown.

Solution

Read some ordered pairs from the graph.

Note that as $x$ changes by 1 unit, $y$ increases by a factor of 3, confirming that this function is an exponential function.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Change in $y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>$\times 3$</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>$\times 3$</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>$\times 3$</td>
</tr>
<tr>
<td>3</td>
<td>54</td>
<td></td>
</tr>
</tbody>
</table>

Since each successive value increases by a factor of 3, this function must have $b = 3$. Since all points on this graph must satisfy the equation $y = ab^x$, substitute the coordinates of one of the points, and the value of $b$, to find the value of $a$.

Pick a point that is easy to work with, such as $(1, 6)$. Substitute $x = 1$, $y = 6$, and $b = 3$:

$y = ab^x$

$6 = a \times 3^1$

$6 = a \times 3$

$a = 2$

Therefore, the equation that describes this curve is $y = 2 \times 3^x$. 
Example 3

Write an Exponential Function Given Its Properties

A radioactive sample has a half-life of 3 days. The initial sample is 200 mg.

a) Write a function to relate the amount remaining, in milligrams, to the time, in days.

b) Restrict the domain of the function so that the mathematical model fits the situation it is describing.

Solution

a) This exponential decay can be modelled using a function of the form

\[ A(x) = A_0 \left( \frac{1}{2} \right)^x \]

where \( x \) is the time, in half-life periods; \( A_0 \) is the initial amount, in milligrams; and \( A \) is the amount remaining, in milligrams, after time \( x \).

Start with 200 mg. After every half-life, the amount is reduced by half.

Substituting \( A_0 = 200 \) into this equation gives \( A(x) = 200 \left( \frac{1}{2} \right)^x \). This expresses \( A \) as a function of \( x \), the number of half-lives. To express \( A \) as a function of \( t \), measured in days, replace \( x \) with \( \frac{t}{3} \).

The half-life of this material is 3 days. Therefore, the number of elapsed half-lives at any given point is the number of days divided by 3.

\[ A(t) = 200 \left( \frac{1}{2} \right)^{\frac{t}{3}} \]

This equation relates the amount, \( A \), in milligrams, of radioactive material remaining to time, \( t \), in days.

b) A graph of this function reveals a limitation of the mathematical model.

The initial sample size, at \( t = 0 \), was 200 mg. It is not clear that the function has any meaning before this time. Since it is only certain that the mathematical model fits this situation for non-negative values of \( t \), it makes sense to restrict its domain:

\[ A(t) = 200 \left( \frac{1}{2} \right)^{\frac{t}{3}} \text{ for } \{ t \in \mathbb{R}, t \geq 0 \} \]
Communicate Your Understanding

C1  a) Is an exponential function either always increasing or always decreasing? Explain.
    b) Is it possible for an exponential function of the form \( y = ab^x \) to have an \( x \)-intercept? If yes, given an example. If no, explain why not.

C2  Consider the exponential functions \( f(x) = 100\left(\frac{1}{2}\right)^x \) and \( g(x) = -10(2)^x \).
    a) Which function has a graph with range
       i) \( y \in \mathbb{R}, y < 0 \)?  
       ii) \( y \in \mathbb{R}, y > 0 \)?
       Explain how you can tell by inspecting the equations.
    b) Which function is
       i) increasing?  
       ii) decreasing?
       Explain how you can tell by inspecting the equations.

C3  Describe what is meant by “asymptotic behaviour.” Support your explanation with one or more sketches.
**A Practise**

For help with questions 1 to 3, refer to Example 1.

1. Match each graph with its corresponding equation.
   - a) $y = 2 \times 2^x$
   - b) $y = 2 \times (\frac{1}{2})^x$
   - c) $y = \frac{1}{2} \times 2^x$
   - d) $y = -2^x$

2. a) Sketch the graph of an exponential function that satisfies all of these conditions:
   - domain \( x \in \mathbb{R} \)
   - range \( y \in \mathbb{R}, y > 0 \)
   - \( y \)-intercept 5
   - function increasing
   b) Is this the only possible graph? Explain.

3. a) Sketch the graph of an exponential function that satisfies all of these conditions:
   - domain \( x \in \mathbb{R} \)
   - range \( y \in \mathbb{R}, y < 0 \)
   - \( y \)-intercept -2
   - function decreasing
   b) Is this the only possible graph? Explain.

For help with questions 4 and 5, refer to Example 2.

4. Write an exponential equation to match the graph shown.
5. Write an exponential equation to match the graph shown.

![Graph of exponential function](image)

For help with question 6, refer to Example 3.

6. A radioactive sample with an initial mass of 25 mg has a half-life of 2 days.
   a) Which equation models this exponential decay, where \( t \) is the time, in days, and \( A \) is the amount of the substance that remains?
   
   \[ A = 25 \times 2^t \]
   
   \[ B = 25 \times \left(\frac{1}{2}\right)^t \]
   
   \[ C = 25 \times \left(\frac{1}{2}\right)^{\frac{1}{2}} \]
   
   \[ D = 2 \times 25^t \]
   
   b) What is the amount of radioactive material remaining after 7 days?

B Connect and Apply

7. Graph each function and identify the
   i) domain
   ii) range
   iii) \( x \)- and \( y \)-intercepts, if they exist
   iv) intervals of increase/decrease
   v) asymptote
   a) \( f(x) = \left(\frac{1}{2}\right)^x \)
   b) \( y = 2 \times 1.5^x \)
   c) \( y = -\left(\frac{1}{3}\right)^x \)

8. a) Graph each function.
   i) \( f(x) = 2^x \)
   ii) \( r(x) = \frac{2}{x} \)
   b) Describe how the graphs are alike. How do they differ?
   c) Compare the asymptotes of these functions. What do you observe?

9. a) Graph each function.
   i) \( g(x) = \left(\frac{1}{2}\right)^x \)
   ii) \( r(x) = \frac{2}{x} \)
   b) Describe how the graphs are alike. How do they differ?
   c) Compare the asymptotes of these functions. What do you observe?

10. a) Predict how the graphs of the following functions are related.
    i) \( f(x) = 3^{-x} \)
    ii) \( g(x) = \left(\frac{1}{3}\right)^x \)
    b) Graph both functions and check your prediction from part a).
    c) Use algebraic reasoning to explain this relationship.

11. The graph shows the voltage drop across a capacitor over time while discharging an RC circuit. At \( t = 0 \) s, the circuit begins to discharge.
   a) What is the domain of this function?
   b) What is the range?
   c) What is the initial voltage drop across the capacitor?
   d) What value does the voltage drop across the capacitor approach as more time passes?
   e) Approximately how long will it take the voltage drop to reach 50% of the initial value?
12. A flywheel is rotating under friction. The number, \( R \), of revolutions per minute after \( t \) minutes can be determined using the function \( R(t) = 4000(0.75)^t \).

a) Explain the roles of the numbers 0.75 and 2 in the equation.

b) Graph the function.

c) Which value in the equation indicates that the flywheel is slowing?

d) Determine the number of revolutions per minute after

i) 1 min

ii) 3 min

C Extend

13. Use Technology Refer to question 11. The equation that models this situation is given by \( V = V_0 b^\frac{t}{RC} \), where \( V \) is the voltage drop, in volts; \( V_0 \) is the initial voltage drop; \( t \) is the time, in seconds; \( R \) is the resistance, in ohms (\( \Omega \)); and \( C \) is the capacitance, in farads (\( F \)).

For this circuit, \( R = 2000 \, \Omega \) and \( C = 1 \, \mu F \).

Note that 1 \( \mu F = 0.000 \, 001 \, F \).

a) Determine the value of the base, \( b \).

b) Explain your method.

c) Graph the function using a graphing calculator or graphing software. Use the window settings shown.

```
WINDOW
xmin=-.01
xmax=1
ymin=-.01
ymax=1

xres=.1
```

d) What are the domain and range of this function?

e) Explain how and why the domain and range are restricted, as illustrated in the graph of question 11.

14. Suppose a square-based pyramid has a fixed height of 25 m.

a) Write an equation, using rational exponents where appropriate, to express the side length of the base of a square-based pyramid in terms of its volume.

b) How should you limit the domain of this function so that the mathematical model fits the situation?

c) What impact does doubling the volume have on the side length of the base? Explain.

15. Suppose that a shelf can hold cylindrical drums with a fixed height of 1 m.

a) Write a simplified equation, using rational exponents where appropriate, to express the surface area in terms of the volume for drums that will fit on the shelf.

b) Find the surface area and diameter of a drum with a volume of 0.8 m\(^3\).

c) What are the restrictions on the domain of the function used in this model?

d) Graph the function for the restricted domain.

16. Math Contest Find all solutions to

\[ 3^x - 2^x - 1 = 0. \]

17. Math Contest Consider the function

\[ y = 12 \left( \frac{1}{2} \right)^x - 3. \]

The y-intercept is \( b \) and the x-intercept is \( a \). The sum of \( a \) and \( b \) is

A 11  B 6  C 7  D 18

18. Math Contest A number is between 20 and 30. When this number is subtracted from its cube, the result is 13 800. When the same number is added to its cube, the answer is

A 13 848  B 13 852  
C 13 846  D 13 844